Solitons in a uniaxial anisotropic Heisenberg spin chain with Gilbert damping in an external magnetic field

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Using the stereographic projection of the unit sphere of the spin field onto a complex plane for the equations of motion in a uniaxial anisotropic Heisenberg spin chain with Gilbert damping in an external magnetic field, the effect of the magnetic field for the integrability of the system is discussed. The effect of the Gilbert damping is also analyzed. Then, introducing an auxiliary parameter, the Lax equations for the Darboux matrices are generated recursively. The Jost solutions satisfy the corresponding Lax equations if constants are suitably chosen. The exact soliton solutions are then investigated. These results show that the solitary waves depend essentially on two velocities that describe a spin configuration deviated from a homogeneous magnetization, while the depths and widths of surface of solitary waves vary periodically with time. The center of an inhomogeneity moves with a constant velocity, while the shape of a soliton also changes with another constant velocity and it is not symmetrical with respect to the center. The z component of the total magnetic momentum and the total magnetic momentum vary with time. The asymptotic behavior of multisoliton solutions is also given. [S1063-651X(97)03002-X]

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I. INTRODUCTION

The classical Heisenberg spin chain is a fascinating nonlinear dynamical system that exhibits both coherent and chaotic structures depending on the nature of the magnetic interactions [1–4]. Its study is of considerable interest, especially from the points of view of soliton theory and condensed-matter physics. In particular, its continuum limit is governed by the Landau-Lifshitz equation and it displays fascinating geometrical aspects: isotropic [5–7] and pure anisotropic [8–13] systems are geometrically equivalent and gauge equivalent to a nonlinear Schrödinger equation [14]. These systems, as well as the biaxial anisotropic [15–17] systems, are completely integrable. On the experimental side, an easy plane ferromagnet in a symmetry-breaking external transverse field has received continuing interest, though most theoretical treatments have been based on an approximate mapping [18] to a sine-Gordon equation.

There are many works in the study of solitons in the classical Heisenberg spin chain. By separating variables in the moving coordinates, Tjon and Wright [19] and Quispel and Capel [20] obtained separately the Landau-Lifshitz equation for an isotropic chain and for a spin chain with an easy axis. Reducing the equation of motion to a sine-Gordon equation for a spin chain with an easy plane, Mikeska [21] got a solution. However, there exist some questions about this approach. First, this reduction has not been rigorously established, except for $T\rightarrow 0$. Second, it does not include the quantum effects [3], which are particularly crucial for the case of $CsNiF_3$ with spin $S=1$. Third, it is inadequate [22,23], as shown by the neutron scattering experiments in $CsNiF_3$. Finally, when the external field tends to zero, this solution becomes a traveling-wave solution, which does not obviously relate to the nonlinearity of the spin interaction. Long and Bishop [24] proposed another solution. However, when the anisotropic approach vanishes this solution does not tend to the well-known solution of an isotropic chain. By the variation method, Nakumura and Sasada [10] obtained a solution. If this solution is directly substituted into the equation of motion, it does not satisfy the equation. Reducing the equation to an appropriate form, Kosevich, Ivanov, and Kovalev [25] found a solution. But it could not be considered as an approximate solution of an equation for a spin chain with an easy plane since it does not satisfy this equation even in the approximation of first-order anisotropy. Borisov [26] and Sklyanin [27] found separately Lax pairs of the equation

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of motion for a spin chain with a complete anisotropy. By means of an inverse scattering transformation, Mikhailov [28] and Rodin [13] were able to reduce the problem to the Riemann boundary-value problem on a torus. However, since these results are expressed by the elliptic function, they are more complicated and are therefore difficult to transform to those in the case of an easy plane. Even though soliton solutions were found, they are difficult to transform to those in the limit of an easy plane. Deriving the Marchenko equation by an inverse scattering transformation, Borovik [29] and Borovik and Kulinich [30] could not find even the single-soliton solution in a ferromagnet with uniaxial anisotropy. Using the Hirota method, Bogdan and Kovalev [31] attempted to construct exact multisoliton solutions in an anisotropic ferromagnet. However, they could not prove a series of nontrivial identities on the parameters of the solution. When the anisotropy of an easy plane was weak, they did not obtain the explicit expressions of the solutions. Taking into account only the first-order approximation, Ivanov, Kosevich, and Babich [32] obtained useful results.

There exist some difficulties in the study of a uniaxial anisotropic Heisenberg chain with Gilbert damping in an external magnetic field. Its equations of motion, which differ from those of an isotropic chain, could not be solved by the method of separating variables in moving coordinates [19,20]. Also, this equation could not be solved by the previous form of an inverse scattering transformation since the double-valued function of the usual spectral parameter appearing here required the introduction of a Riemann surface. The reflection coefficient at the edges of cuts in the complex plane could not be neglected even in the case of nonreflection. Introducing an auxiliary parameter, Chen, Huang, and Liu [33] developed an inverse scattering transformation to solve the Landau-Lifshitz equation only for a spin chain with an easy axis. The Marchenko equation, soliton solutions, and asymptotic behavior were derived. The results can reduce naturally to those of an isotropic chain when the anisotropy vanishes. By means of the method of Darboux transformation [34–38], Huang, Chen, and Liu [39] found the exact soliton solutions of the Landau-Lifshitz equation for a spin chain only with an easy plane without an external magnetic field. The external magnetic field would affect the integrability of the system. It would be instructive if the effect of the magnetic field is discussed. Pu, Zhou, and Li [40] reported the multisoliton solutions of the Landau-Lifshitz equation in an isotropic ferromagnetic chain with a magnetic field. By means of the Holstein-Primakoff transformation and Glauber's coherent state representation, Huang *et al.* [41,42] and Shi *et al.* [43] used an inhomogeneous Heisenberg spin Hamiltonian with single ion anisotropy in a magnetic field to investigate the nonlinear excitations in a ferromagnetic chain. They reduced the equation of motion into a nonlinear Schrödinger equation. Then, in terms of an inverse scattering transformation, they obtained the corresponding single-soliton and two-magnon bound-state solutions in a homogeneous system. Introducing the coherent-state ansatz, the time-dependent variational principle, and the method of multiple scales, Liu and Zhou reduced the equation of motion into a nonlinear Schrödinger equation and obtained solitons in the pure [44] and the biaxial [45] anisotropic antiferromagnetic spin chains with an external field. We wish to show the astonishing fact that the effect of Gilbert damping [46–48] is just a rescaling of the time variable t by a complex constant, so that for every given solution of the undamped equations of motion in any dimension the exact solution of the fully damped version can be given straightforwardly.

It is the purpose of this paper to investigate the exact soliton solutions in a uniaxial anisotropic Heisenberg spin chain with Gilbert damping in an external magnetic field. This article is organized as follows. In Sec. II using the stereographic projection of the unit sphere of the spin field onto a complex plane for the equations of motion, the effect of the magnetic field for integrability of the system is discussed and the effect of the Gilbert damping is analyzed. Then, introducing an auxiliary parameter, the Lax equations for the Darboux matrices are generated recursively. Section III shows that the Jost solutions satisfy the corresponding Lax equations if constants are suitably chosen. The exact soliton solutions, the z component of the total magnetic momentum, and the total magnetic momentum are obtained. In Sec. IV the asymptotic behavior of multisoliton solutions is also given. Section V is discussion. This approach is a good method in the study of solitons in the classical Heisenberg spin chains.

II. EQUATIONS OF MOTION

The Hamiltonian describing a uniaxial anisotropic Heisenberg spin chain with Gilbert damping in an external magnetic field $\mathbf{B}(t)$ can be written as

$$\begin{aligned} H = & -J\sum_i \mathbf{S}_i\cdot\mathbf{S}_{i+1} + A\sum_i (S_i^z)^2 - g\mu_B\sum_i \mathbf{B}\cdot\mathbf{S}_i \\ & + \epsilon\mathbf{S}_i\cdot\left[-J\sum_i \mathbf{S}_i\cdot\mathbf{S}_{i+1} + A\sum_i (S_i^z)^2 - g\mu_B\sum_i \mathbf{B}\cdot\mathbf{S}_i\right], \end{aligned} \tag{1}$$

where $\mathbf{S}_i=(S_i^x,S_i^y,S_i^z)$, with $i=1,2,\ldots,N$ are three-component unit vectors ($|\mathbf{S}|=1$) with only nearest-neighbor interactions, $J>0$ is the pair interaction parameter, A is a uniaxial anisotropic parameter ($A>0$, easy plane; $A<0$, easy axis), g is the Lande factor, μ_B is the Bohr magneton, $\mathbf{B}(t)=(B^x(t),B^y(t),B^z(t))$, and ϵ is a dimensionless Gilbert damping parameter. Using the suitable rescaling and an appropriate spin Poisson bracket, the corresponding equation of motion in the continuum limit can be written as

$$\begin{aligned} \partial_t\mathbf{S} = & \mathbf{S}\times[J\partial_{xx}\mathbf{S} - 2A(\mathbf{S}\cdot\mathbf{n})\mathbf{n} + g\mu_B\mathbf{B}] \\ & + \epsilon\mathbf{S}\times\mathbf{S}\times[J\partial_{xx}\mathbf{S} - 2A(\mathbf{S}\cdot\mathbf{n})\mathbf{n} + g\mu_B\mathbf{B}], \end{aligned} \tag{2}$$

where $\mathbf{S}(x,t)=(S^x,S^y,S^z)$, $\mathbf{S}^2(x,t)=1$, and $\mathbf{n}=(0,0,1)$. In the undamped case ($\epsilon=0$), when an external magnetic field is zero, an anisotropic spin chain ($A\neq 0$) with an easy plane ($A>0$) [39] and that with an easy axis ($A<0$) [33] are completely integrable. When the oscillations of the spin vector $\mathbf{S}$ are localized near an easy plane, Eq. (2) can be transformed into a sine-Gordon equation. Similarly, it is also reduced to a nonlinear Schrödinger equation when the oscillations of the spin vector $\mathbf{S}$ are localized in the vicinity of the vacuum state $\mathbf{S}(x,t)=(0,0,1)$. In the special case

$A=0$, an isotropic spin chain in an external magnetic field is also completely integrable [40]. When a magnetic field is zero, Eq. (2) is equivalent to a nonlinear Schrödinger equation [5–7].

We first consider the effect of the Gilbert damping term, proportional to ϵ in Eqs. (1) or (2), on undamped spin motion. Traditional treatments for Eqs. (1) or (2) in polar coordinates tend to mix up the evolutions of the two angles in a complicated way, so the Gilbert damping is treated only approximately [48]. However, now the parametrization of the spin field in terms of a stereographic variable simplifies the structure of Eqs. (1) or (2) drastically. Using stereographic projection of the unit sphere of the spin field onto a complex plane

$$P(x,t)=\frac{S^x+iS^y}{1+S^z} \tag{3}$$

or

$$S^x+iS^y=\frac{2P}{1+|P|^2},\quad S^z=\frac{1-|P|^2}{1+|P|^2}, \tag{4}$$

the derivatives can be written as

$$\partial_t S^x=\frac{1}{(1+|P|^2)^2}[(1-P^{*2})\partial_t P+(1-P^2)\partial_t P^*], \tag{5}$$

$$\partial_t S^y=-\frac{i}{(1+|P|^2)^2}[(1+P^{*2})\partial_t P-(1+P^2)\partial_t P^*], \tag{6}$$

$$\partial_t S^z=-\frac{2}{(1+|P|^2)^2}[P^*\partial_t P+P\partial_t P^*], \tag{7}$$

and

$$\begin{aligned}\partial_{xx}S^x=&\frac{1}{(1+|P|^2)^2}[(1-P^{*2})\partial_{xx}P+(1-P^2)\partial_{xx}P^*]\\&-\frac{2}{(1+|P|^2)^3}[2(P+P^*)\partial_x P\partial_x P^*+P^*(1-P^{*2})\\&\times(\partial_x P)^2+P(1-P^2)(\partial_x P^*)^2],\end{aligned} \tag{8}$$

$$\begin{aligned}\partial_{xx}S^y=&-\frac{i}{(1+|P|^2)^2}[(1+P^{*2})\partial_{xx}P-(1+P^2)\partial_{xx}P^*]\\&+\frac{2i}{(1+|P|^2)^3}[2(P-P^*)\partial_x P\partial_x P^*+P^*(1+P^{*2})\\&\times(\partial_x P)^2-P(1+P^2)(\partial_x P^*)^2],\end{aligned} \tag{9}$$

$$\begin{aligned}\partial_{xx}S^z=&\frac{1}{(1+|P|^2)^3}[P^*(1+|P|^2)\partial_{xx}P+P(1+|P|^2)\partial_{xx}P^*\\&+2(1-|P|^2)\partial_x P\partial_x P^*-2P^{*2}(\partial_x P)^2\\&-2P^2(\partial_x P^*)^2].\end{aligned} \tag{10}$$

Substituting Eqs. (4)–(10) into the equations of three components of Eq. (2),

$$\begin{aligned}\partial_t S^x=&(S^y\partial_{xx}S^z-S^z\partial_{xx}S^y)-2AS^yS^z+g\mu_B S^yB^z\\&+\epsilon[\partial_{xx}S^x-(S^x\partial_{xx}S^x-2A(S^z)^2+g\mu_B B^x)S^x],\end{aligned} \tag{11}$$

$$\begin{aligned}\partial_t S^y=&(S^z\partial_{xx}S^x-S^x\partial_{xx}S^z)-2AS^zS^x+g\mu_B S^zB^x\\&+\epsilon[\partial_{xx}S^y-(S^y\partial_{xx}S^y-2A(S^z)^2+g\mu_B B^y)S^y],\end{aligned} \tag{12}$$

$$\begin{aligned}\partial_t S^z=&(S^x\partial_{xx}S^y-S^y\partial_{xx}S^x)-2AS^xS^y+g\mu_B S^xB^y\\&+\epsilon[\partial_{xx}S^z-(S^z\partial_{xx}S^z-2A(S^z)^2+g\mu_B B^z)S^z],\end{aligned} \tag{13}$$

we can obtain

$$\begin{aligned}&(1-P^{*2})\Phi(P,P^*)-(1-P^2)\Phi^*(P,P^*)=0,\\&-i(1+P^{*2})\Phi(P,P^*)-i(1+P^2)\Phi^*(P,P^*)=0,\\&P^*\Phi(P,P^*)-P\Phi^*(P,P^*)=0,\end{aligned} \tag{14}$$

where

$$\begin{aligned}\Phi(P,P^*)=&i(1+|P|^2)\partial_t P+(1-i\epsilon)\Big\{(1+|P|^2)\partial_{xx}P\\&-2P^*(\partial_x P)^2+2AP(1-|P|^2)+g\mu_B(1+|P|^2)\\&\times\left[\frac{B^x}{2}(1-P^2)+i\frac{B^y}{2}(1+P^2)-B^zP\right]\Big\}.\end{aligned} \tag{15}$$

The consistency of Eq. (14) implies $\Phi(P,P^*)=0$ and $\Phi^*(P,P^*)=0$; therefore, the evolution equation for the stereographic variable $P(x,t)$ in the presence of the Gilbert damping becomes

$$\begin{aligned}&i(1+|P|^2)\partial_t P+(1-i\epsilon)\Big\{(1+|P|^2)\partial_{xx}P-2P^*(\partial_x P)^2\\&+2AP(1-|P|^2)+g\mu_B(1+|P|^2)\\&\times\left[\frac{B^x}{2}(1-P^2)+i\frac{B^y}{2}(1+P^2)-B^zP\right]\Big\}=0.\end{aligned} \tag{16}$$

If the time variable is redefined

$$t\to\tau=(1-i\epsilon)t, \tag{17}$$

we can obtain

$$\begin{aligned}&i(1+|P|^2)\partial_\tau P+(1+|P|^2)\partial_{xx}P-2P^*(\partial_x P)^2\\&\quad+2AP(1-|P|^2)+g\mu_B(1+|P|^2)\\&\quad\times\left[\frac{B^x}{2}(1-P^2)+i\frac{B^y}{2}(1+P^2)-B^zP\right]=0,\end{aligned} \tag{18}$$

which is the same as the undamped evolution equation for P endowed here with the scaled time τ. Therefore, as long as every solution in the undamped case ($\epsilon=0$) is obtained, the corresponding spin field $\mathbf{S}(x,t)$ in the Gilbert damping case

$(\epsilon \neq 0)$ can be constructed simply by Eq. (4) just with the rescaling in Eq. (18) of the time parameter.

According to Eq. (18), we can also analyze the effect of an external magnetic field for the integrability of the system. When an external field is directed at an anisotropic axis, e.g., $\mathbf{B}=(0,0,B^z(t))$, the magnetic-field term in Eq. (18) can be removed by the following gauge transformation:

$$P \to \widetilde{P} = P \, \exp\left[i g \mu_B \int d\tau B^z(\tau) \right]; \tag{19}$$

the system becomes integrable. The influence of the magnetic field for the classical Heisenberg spin chain with an easy axis amounts to a change of the precession frequency of the spin field $\mathbf{S}$ by $\omega_B = g\mu_B B$. Therefore, if we can introduce a new angular variable $\widetilde{\varphi} = \varphi - \omega_B t$ in the polar coordinates (θ, φ), then in terms of the angular variables θ and $\widetilde{\varphi}$ the equation of motion (2) will not depend on B.

However, the dynamics of the classical Heisenberg spin chain with an easy plane is very sensitive to a magnetic field. Even a weak magnetic field can alter the character of the ground state and therefore the form of localized solutions. When an external magnetic field is perpendicular to an easy plane, it does not alter the axial symmetry associated with the z axis; the form of the ground state depends on the strength of the external field. The critical value is $B_c = 2AS/g\mu_B$. When the external magnetic field $B^z < B_c$, the spin field $\mathbf{S}$ in the ground state deviates from an easy plane and it is characterized by an inclination $\theta = \theta_0$ to the z axis, where $\cos\theta_0 = B^z/B_c$. The angle φ remains arbitrary. For brevity, such a ground state is referred to as an easy cone. As an external magnetic field increases, the angular opening of the easy cone becomes smaller, especially in the case of $B^z \gg B_c$; the spin field $\mathbf{S}$ in a nonexcited Heisenberg spin chain with an easy plane lies along the z axis.

In the context of experiments [22,23], the situation where the external magnetic field lies in an easy plane, e.g., $\mathbf{B}=(B^x(t),0,0)$, or $\mathbf{B}=(0,B^y(t),0)$, seems quite typical. In experiments on samples of easy plane ferromagnets $CsNiF_3$ and $(C_6H_{11}N\,H_3)Cu\,Br_3$ an external field is applied as a rule in an easy plane [18,21]. The presence of an external field that lies in an easy plane makes finding soliton solutions of the equation of motion (2) more difficult. The magnetic-field term in Eq. (18) is not removable by gauge transformation (19) and none of the spin components remain conserved quantities. Consequently, the combined Galilean plus gauge invariance of the equation of motion is broken and no Lax pairs seem to exist; the system appears to be nonintegrable.

Equation (2) may be represented as a compatibility condition $\partial_t L - \partial_x M + [L,M] = 0$ of two equations for 2×2 matrices $\Psi(x,t;\mu,\lambda)$:

$$\begin{aligned} \partial_x \Psi(x,t;\mu,\lambda) &= L(\mu,\lambda)\Psi(x,t;\mu,\lambda), \\ \partial_t \Psi(x,t;\mu,\lambda) &= M(\mu,\lambda)\Psi(x,t;\mu,\lambda), \end{aligned} \tag{20}$$

while

$$L(\mu,\lambda) = -i\mu(S^x\sigma_x + S^y\sigma_y) - i\lambda S^z\sigma_z \tag{21}$$

and

$$\begin{aligned} M(\mu,\lambda) = {} & i2\mu\lambda(S^x\sigma_x + S^y\sigma_y) + i2\mu^2 S^z\sigma_z - i\mu(S^y\partial_x S^z \\ & - S^z\partial_x S^y)\sigma_x - i\mu(S^z\partial_x S^x - S^x\partial_x S^z)\sigma_y \\ & - i\lambda(S^x\partial_x S^y - S^y\partial_x S^x)\sigma_z, \end{aligned} \tag{22}$$

where σ_α $(\alpha = x,y,z)$ are the Pauli metrics and the parameters λ and μ satisfy the relation

$$\lambda^2 = \begin{cases} \mu^2 + 4\rho^2 & \text{for } A>0 \ \ (\text{easy plane}) \\ \mu^2 - 4\rho^2 & \text{for } A<0 \ \ (\text{easy axis}), \end{cases} \tag{23}$$

where ρ is defined as

$$\rho = \begin{cases} \left(\dfrac{2AS - g\mu_B B}{4JS} \right)^{1/2} & \text{for } A>0 \ \ (\text{easy plane}) \\ \left(\left| \dfrac{2AS + g\mu_B B}{4JS} \right| \right)^{1/2} & \text{for } A<0 \ \ (\text{easy axis}). \end{cases} \tag{24}$$

If one of the parameters in Eq. (23) is taken as an independent parameter, then the others are the double-value functions of the first, and it is then necessary to introduce a Riemann surface. In order to avoid the complexity brought about by a Riemann surface, we shall introduce an auxiliary parameter ξ,

$$\lambda = \begin{cases} \xi + \rho^2\xi^{-1} & \text{for an easy plane} \\ \xi - \rho^2\xi^{-1} & \text{for an easy axis} \end{cases} \tag{25}$$

and

$$\mu = \begin{cases} \xi - \rho^2\xi^{-1} & \text{for an easy plane} \\ \xi + \rho^2\xi^{-1} & \text{for an easy axis,} \end{cases} \tag{26}$$

where $\xi = \pm\rho$ corresponds to zero μ (or λ) and to $\lambda = \pm 2\rho$ (or $\mu = \pm 2\rho$). In the complex λ (or μ) plane, these two points are the edges of cuts. This indicates that in an inverse scattering transformation the edges of cuts must give a contribution even in the case of nonreflection.

For transformations (25) and (26) we have the relations

$$\begin{aligned} \lambda(-\overline{\xi}) &= -\overline{\lambda(\xi)}, \\ \mu(-\overline{\xi}) &= -\overline{\mu(\xi)}. \end{aligned} \tag{27}$$

The corresponding Lax equations are written as

$$\begin{aligned} \partial_x \Psi(\xi) &= L(\xi)\Psi(\xi), \\ \partial_t \Psi(\xi) &= M(\xi)\Psi(\xi), \end{aligned} \tag{28}$$

where

$$L(\xi)=\begin{cases} -i(\xi-\rho^2\xi^{-1})(S^x\sigma_x+S^y\sigma_y)-i(\xi+\rho^2\xi^{-1})S^z\sigma_z & \text{for an easy plane} \\ -i(\xi+\rho^2\xi^{-1})(S^x\sigma_x+S^y\sigma_y)-i(\xi-\rho^2\xi^{-1})S^z\sigma_z & \text{for an easy axis} \end{cases} \tag{29}$$

and

$$\begin{aligned} M(\xi)=&\,i2(\xi^2-\rho^4\xi^{-2})(S^x\sigma_x+S^y\sigma_y)+i2(\xi-\rho^2\xi^{-1})^2S^z\sigma_z \\ &-i(\xi-\rho^2\xi^{-1})(S^y\partial_xS^z-S^z\partial_xS^y)\sigma_x \\ &-i(\xi-\rho^2\xi^{-1})(S^z\partial_xS^x-S^x\partial_xS^z)\sigma_y \\ &-i(\xi-\rho^2\xi^{-1})(S^x\partial_xS^y-S^y\partial_xS^x)\sigma_z \end{aligned} \tag{30}$$

for an easy plane and

$$\begin{aligned} M(\xi)=&\,i2(\xi^2-\rho^4\xi^{-2})(S^x\sigma_x+S^y\sigma_y)+i2(\xi+\rho^2\xi^{-1})^2S^z\sigma_z \\ &-i(\xi+\rho^2\xi^{-1})(S^y\partial_xS^z-S^z\partial_xS^y)\sigma_x \\ &-i(\xi+\rho^2\xi^{-1})(S^z\partial_xS^x-S^x\partial_xS^z)\sigma_y \\ &-i(\xi+\rho^2\xi^{-1})(S^x\partial_xS^y-S^y\partial_xS^x)\sigma_z \end{aligned} \tag{31}$$

for an easy axis. We can find the relations

$$L(-\overline{\xi})=\begin{cases} \sigma_x\overline{L(\xi)}\sigma_x & \text{for an easy plane} \\ \sigma_z\overline{L(\xi)}\sigma_z & \text{for an easy axis,} \end{cases} \tag{32}$$

$$L^\dagger(\overline{\xi})=\begin{cases} -L(\xi) & \text{for an easy plane} \\ -L(\xi) & \text{for an easy axis,} \end{cases} \tag{33}$$

$$M(-\overline{\xi})=\begin{cases} \sigma_x\overline{M(\xi)}\sigma_x & \text{for an easy plane} \\ \sigma_z\overline{M(\xi)}\sigma_z & \text{for an easy axis,} \end{cases} \tag{34}$$

and

$$M^\dagger(\overline{\xi})=\begin{cases} -M(\xi) & \text{for an easy plane} \\ -M(\xi) & \text{for an easy axis.} \end{cases} \tag{35}$$

There are two different types of physical boundary conditions in Eq. (2). The boundary condition of the first type corresponds to a breatherlike solution, which is usually called a magnetic soliton. In the classical Heisenberg spin chain with an easy plane in an external magnetic field, the spin field S in the ground state deviates from an easy plane and it is characterized by an inclination θ_0 to the z axis and the asymptotic spin lies on the surface of an easy cone. The simplest solution of Eq. (2) can be written as

$$\mathbf{S}=\mathbf{S}_0=(S_0\sin\theta_0,\ 0,\ S_0\cos\theta_0). \tag{36}$$

The corresponding Jost solution of Eq. (28) may be chosen as

$$\begin{aligned} \Psi_0(\xi)=&\frac{1}{2}[I-i(\sigma_x+\sigma_y+\sigma_z)]\exp\Big\{-iS_0\sin\theta_0(\xi-\rho^2\xi^{-1}) \\ &\times[x-2(\xi+\rho^2\xi^{-1})t]\sigma_x-iS_0\cos\theta_0(\xi+\rho^2\xi^{-1}) \\ &\times\left[x-2\frac{(\xi^2-\rho^2)^2}{\xi(\xi^2+\rho^2)}t\right]\sigma_z\Big\}. \end{aligned} \tag{37}$$

Since the z axis is an easy axis in the classical Heisenberg spin chain with an easy axis, the boundary condition is chosen as

$$S\to S_0=(0,0,S_0) \quad \text{at} \quad x\to\pm\infty. \tag{38}$$

The corresponding Jost solution of Eq. (28) may be chosen as

$$\begin{aligned} \Psi_0(\xi)=&\frac{1}{2}[I-i(\sigma_x+\sigma_y+\sigma_z)] \\ &\times\exp\left\{-iS_0(\xi-\rho^2\xi^{-1})\left[x-2\frac{(\xi^2+\rho^2)^2}{\xi(\xi^2-\rho^2)}t\right]\sigma_z\right\}, \end{aligned} \tag{39}$$

with the relations

$$\Psi_0(-\overline{\xi})=\begin{cases} -i\sigma_x\overline{\Psi_0(\xi)} & \text{for an easy plane} \\ -i\sigma_z\overline{\Psi_0(\xi)} & \text{for an easy axis} \end{cases} \tag{40}$$

and

$$\Psi_0^\dagger(\overline{\xi})=\begin{cases} \Psi_0^{-1}(\xi) & \text{for an easy plane} \\ \Psi_0^{-1}(\xi) & \text{for an easy axis.} \end{cases} \tag{41}$$

The method of Darboux transformation [34–39] is one of the most powerful methods for constructing exact solutions of nonlinear integrable systems. In the rest of this paper, we will use the Darboux matrices $D_n(\xi)$ to define the Jost solution $\Psi_n(\xi)$ of Eq. (28) such that

$$\Psi_n(\xi)=D_n(\xi)\Psi_{n-1}(\xi), \tag{42}$$

where $n=1,2,3,\ldots$ and $D_n(\xi)$ has two poles ξ_n and $-\overline{\xi}_n$.

Substituting Eq. (42) into Eq. (28) with a suitable subscript, the Lax equations for $D_n(\xi)$ can be written as

$$\begin{aligned} \partial_xD_n(\xi)&=L_n(\xi)D_n(\xi)-D_n(\xi)L_{n-1}(\xi), \\ \partial_tD_n(\xi)&=M_n(\xi)D_n(\xi)-D_n(\xi)M_{n-1}(\xi). \end{aligned} \tag{43}$$

Then, using the previous relations, we can find

$$\Psi_n(-\overline{\xi})=\begin{cases} -i\sigma_x\overline{\Psi_n(\xi)} & \text{for an easy plane} \\ -i\sigma_z\overline{\Psi_n(\xi)} & \text{for an easy axis,} \end{cases} \tag{44}$$

$$\Psi_n^\dagger(\overline{\xi})=\begin{cases}\Psi_n^{-1}(\xi) & \text{for an easy plane}\\ \Psi_n^{-1}(\xi) & \text{for an easy axis,}\end{cases}\tag{45}$$

$$D_n(-\overline{\xi})=\begin{cases}\sigma_x\overline{D_n(\xi)}\sigma_x & \text{for an easy plane}\\ \sigma_z\overline{D_n(\xi)}\sigma_z & \text{for an easy axis,}\end{cases}\tag{46}$$

and

$$D_n^\dagger(\overline{\xi})=\begin{cases}D_n^{-1}(\xi) & \text{for an easy plane}\\ D_n^{-1}(\xi) & \text{for an easy axis.}\end{cases}\tag{47}$$

When $D_n(\xi)$ has only two simple poles ξ_n and $-\overline{\xi}_n$, one can define

$$D_n(\xi)=C_nB_n(\xi),$$
$$D_n^\dagger(\overline{\xi})=B_n^\dagger(\overline{\xi})C_n^\dagger,\tag{48}$$
$$D_n^{-1}(\xi)=B_n^{-1}(\xi)C_n^{-1},$$

where

$$B_n(\xi)=I-\frac{\xi_n-\overline{\xi}_n}{\xi_n-\xi}F_n-\frac{\overline{\xi}_n-\xi_n}{\overline{\xi}_n+\xi}\widetilde{F}_n,\tag{49}$$

C_n, F_n, and $\widetilde{F}_n$ are 2×2 matrices independent of ξ, and

$$(\xi_n-\overline{\xi}_n)C_nF_n,\quad (\xi_n-\overline{\xi}_n)C_n\widetilde{F}_n\tag{50}$$

are residues at poles ξ_n and $-\overline{\xi}_n$, respectively.

III. SOLITONS

In this section, we will determine separately C_n and $B_n(\xi)$. By means of Eqs. (46)–(48) for C_n, we can obtain the relations

$$C_n=\begin{cases}\sigma_x\overline{C_n}\sigma_x & \text{for an easy plane}\\ \sigma_z\overline{C_n}\sigma_z & \text{for an easy axis}\end{cases}\tag{51}$$

and

$$C_n^\dagger=C_n^{-1},$$
$$C_nC_n^\dagger=I.\tag{52}$$

This shows that C_n is a diagonal, i.e.,

$$(C_n)_{12}=(C_n)_{21}=0,$$
$$(C_n)_{11}=\overline{(C_n)_{22}},\tag{53}$$
$$|(C_n)_{11}|=1.$$

Since only the module of $(C_n)_{11}$ is equal to 1, one can write

$$C_n=\begin{cases}\exp\left(\frac{i}{2}\phi_n\sigma_z\right) & \text{for an easy plane}\\ \exp\left(\frac{i}{2}\phi_n\sigma_z\right) & \text{for an easy axis,}\end{cases}\tag{54}$$

where ϕ_n is real and characterizes the rotation angle of spin in an easy plane; while $\exp[(i/2)\phi_n\sigma_z]$ is a rotation around an easy axis, it does not affect the value of S_z.

In order to determine C_n, substituting Eq. (48) into Eq. (43) and then taking the limits $\xi\to\infty$ and 0, we can obtain

$$\partial_xC_n=-i2\rho(S_n)_z\sigma_zC_n+C_ni2\rho(S_n)_z\sigma_z,$$
$$\partial_x[C_nB_n(0)]=i2\rho(S_n)_z\sigma_z[C_nB_n(0)]-[C_nB_n(0)]i2\rho(S_n)_z\sigma_z\tag{55}$$

for an easy plane and

$$\partial_xC_n=-i2\rho(S_n)_x\sigma_xC_n+C_ni2\rho(S_n)_x\sigma_x,$$
$$\partial_x[C_nB_n(0)]=i2\rho(S_n)_x\sigma_x[C_nB_n(0)]-[C_nB_n(0)]i2\rho(S_n)_x\sigma_x\tag{56}$$

for an easy axis. Comparing these two equations, one can find

$$C_n^{-2}=B_n(0).\tag{57}$$

Therefore, we can obtain C_n as long as $B_n(0)$ is determined.

In terms of Eqs. (46)–(48) for $B_n(\xi)$, we can also obtain the relations

$$\widetilde{B}_n(\xi)=\begin{cases}\sigma_x\overline{B_n(\xi)}\sigma_x & \text{for an easy plane}\\ \sigma_z\overline{B_n(\xi)}\sigma_z & \text{for an easy axis}\end{cases}\tag{58}$$

and

$$B_n^\dagger(\overline{\xi})=B_n^{-1}(\xi),\tag{59}$$

where

$$B_n^\dagger(\overline{\xi})=I-\frac{\overline{\xi}_n-\xi_n}{\overline{\xi}_n-\xi}F_n^\dagger-\frac{\xi_n-\overline{\xi}_n}{\xi_n+\xi}\sigma_xF_n^T\sigma_x\tag{60}$$

for an easy plane and

$$B_n^\dagger(\overline{\xi})=I-\frac{\overline{\xi}_n-\xi_n}{\overline{\xi}_n-\xi}F_n^\dagger-\frac{\xi_n-\overline{\xi}_n}{\xi_n+\xi}\sigma_zF_n^T\sigma_z\tag{61}$$

for an easy axis, where the superscript T means transpose. By means of Eq. (48), $D_n(\xi)D_n^{-1}(\xi)=D_n^{-1}(\xi)D_n(\xi)=I$; it has not poles, i.e., $F_nB_n^\dagger(\xi_n)=0$,

$$F_n\left(I-F_n^\dagger-\frac{\xi_n-\overline{\xi}_n}{2\xi_n}\sigma_xF_n^T\sigma_x\right)=0\tag{62}$$

for an easy plane and

$$F_n\left(I-F_n^\dagger-\frac{\xi_n-\overline{\xi}_n}{2\xi_n}\sigma_zF_n^T\sigma_z\right)=0\tag{63}$$

for an easy axis. This shows that F_n is degenerate. Setting

$$F_n=(\alpha_n\beta_n)^T(\gamma_n\delta_n)\tag{64}$$

and then substituting it into Eqs. (62) and (63), we can obtain the linear equations

$$\gamma_n-(|\gamma_n|^2+|\delta_n|^2)\overline{\alpha_n}-\frac{\xi_n-\overline{\xi_n}}{\xi_n}\gamma_n\delta_n\beta_n=0,$$

$$\delta_n-(|\gamma_n|^2+|\delta_n|^2)\overline{\beta_n}-\frac{\xi_n-\overline{\xi_n}}{\xi_n}\gamma_n\delta_n\alpha_n=0. \qquad (65)$$

Using γ_n and δ_n to express α_n and β_n, F_n and $\widetilde{F}_n$ can be expressed by

$$F_n=\frac{\xi}{\Delta_n}\begin{pmatrix}\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2 & 0\\ 0 & \overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2\end{pmatrix}\begin{pmatrix}\overline{\gamma_n}\\ \overline{\delta_n}\end{pmatrix}(\gamma_n\delta_n) \qquad (66)$$

and

$$\widetilde{F}_n=\frac{\overline{\xi}}{\Delta_n}\begin{pmatrix}\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2 & 0\\ 0 & \overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2\end{pmatrix}\begin{pmatrix}\gamma_n\\ \delta_n\end{pmatrix}(\overline{\gamma_n}\,\overline{\delta_n}), \qquad (67)$$

where

$$\Delta_n=(\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2)(\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2). \qquad (68)$$

Substituting Eqs. (66) and (67) into Eq. (49), $B_n(\xi)$ can be written as

$$\begin{aligned}B_n(\xi)=&\frac{1}{(\xi-\xi_n)(\xi+\overline{\xi_n})\Delta_n}\\ &\times\begin{pmatrix}\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2 & 0\\ 0 & \overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2\end{pmatrix}\\ &\times\Bigg[\xi^2\begin{pmatrix}\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2 & 0\\ 0 & \overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2\end{pmatrix}\\ &+\xi(\xi_n^2-\overline{\xi_n^2})\begin{pmatrix}0 & \overline{\gamma_n}\delta_n\\ \overline{\delta_n}\gamma_n & 0\end{pmatrix}-|\xi_n|^2\\ &\times\begin{pmatrix}\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2 & 0\\ 0 & \overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2\end{pmatrix}\Bigg].\end{aligned} \qquad (69)$$

Using Eqs. (57) and (69), C_n can be determined by

$$C_n=\frac{1}{\delta_n^{1/2}}\begin{pmatrix}\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2 & 0\\ 0 & \overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2\end{pmatrix}, \qquad (70)$$

while ϕ_n in Eq. (54) can be written as

$$\phi_n=2\tan^{-1}\left[\frac{\xi_n''(|\gamma_n|^2-|\delta_n|^2)}{\xi_n'(|\gamma_n|^2+|\delta_n|^2)}\right], \qquad (71)$$

where ξ_n' and ξ_n'' denote the real and imaginary part of ξ_n, respectively.

Up to now, we have obtained C_n and $B_n(\xi)$, i.e., the Darboux matrices $D_n(\xi)$ have been recursively determined. Substituting Eqs. (69) and (70) into Eq. (48), $D_n(\xi)$ can be written as

$$\begin{aligned}D_n(\xi)=&\frac{1}{(\xi-\xi_n)(\xi+\overline{\xi_n})\Delta_n^{3/2}}\begin{pmatrix}(\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2)(\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2) & 0\\ 0 & (\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2)(\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2)\end{pmatrix}\\ &\times\Bigg[\xi^2\begin{pmatrix}\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2 & 0\\ 0 & \overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2\end{pmatrix}+\xi(\xi_n^2-\overline{\xi_n^2})\begin{pmatrix}0 & \overline{\gamma_n}\delta_n\\ \overline{\delta_n}\gamma_n & 0\end{pmatrix}\\ &-|\xi_n|^2\begin{pmatrix}\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2 & 0\\ 0 & \overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2\end{pmatrix}\Bigg].\end{aligned} \qquad (72)$$

In order to determine γ_n and δ_n, substituting Eq. (48) into Eq. (43) and then taking the limit $\xi\to\xi_n$, Eq. (43) can be rewritten as

$$\partial_x[C_nF_n\Psi_{n-1}(\xi_n)]=L_n(\xi_n)C_nF_n\Psi_{n-1}(\xi_n),\quad \partial_t[C_nF_n\Psi_{n-1}(\xi_n)]=M_n(\xi_n)C_nF_n\Psi_{n-1}(\xi_n), \qquad (73)$$

where the factor is independent of x and t. Because F_n is degenerate, the second factor on the right-hand side, i.e., $(\gamma_n\delta_n)\Psi_{n-1}(\xi_n)$, should appear on the left-hand side with its original form, therefore, we can simply obtain

$$(\gamma_n\delta_n)=(b_n 1)\Psi_{n-1}^{-1}(\xi_n), \qquad (74)$$

where b_n is a constant that will be determined by the boundary condition and the initial condition. When $\xi\to 1$, according to Eqs. (43)–(47), we can obtain

$$(\mathbf{S}_n\cdot\boldsymbol{\sigma})=D_n(1)(\mathbf{S}_{n-1}\cdot\boldsymbol{\sigma})D_n^{\dagger}(1), \qquad (75)$$

where $D_n(1)$ can be written as

$$D_n(1)=\frac{1}{(1-\xi_n)(1+\overline{\xi_n})\Delta_n^{1/2}}\begin{pmatrix}(\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2)(\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2) & 0\\ 0 & (\overline{\xi_n}|\gamma_n|^2+\xi_n|\delta_n|^2)(\overline{\xi_n}|\delta_n|^2+\xi_n|\gamma_n|^2)\end{pmatrix}$$
$$\times\begin{pmatrix}(1-\xi_n^2)\overline{\xi_n}|\delta_n|^2+(1-\overline{\xi_n^2})\xi_n|\gamma_n|^2 & (\xi_n^2-\overline{\xi_n^2})\overline{\gamma_n}\delta_n\\ (\xi_n^2-\overline{\xi_n^2})\overline{\delta_n}\gamma_n & (1-\xi_n^2)\overline{\xi_n}|\gamma_n|^2+(1-\overline{\xi_n^2})\xi_n|\delta_n|^2\end{pmatrix}. \tag{76}$$

Similarly, by means of Eqs. (43)–(47), we can also obtain the relations for $\xi\to-1$,

$$\sigma_z(\mathbf{S}_n\cdot\boldsymbol{\sigma})\sigma_z=D_n(-1)\sigma_z(\mathbf{S}_{n-1}\cdot\boldsymbol{\sigma})\sigma_z D_n^{\dagger}(-1) \tag{77}$$

for an easy plane and

$$\sigma_x(\mathbf{S}_n\cdot\boldsymbol{\sigma})\sigma_x=D_n(-1)\sigma_x(\mathbf{S}_{n-1}\cdot\boldsymbol{\sigma})\sigma_x D_n^{\dagger}(-1) \tag{78}$$

for an easy axis. Using Eqs. (43)–(47) and $C_nC_n^{\dagger}=I$ in Eq. (52), Eqs. (77) and (78) can be rewritten as

$$\sigma_z(\mathbf{S}_n\cdot\boldsymbol{\sigma})\sigma_z=-\sigma_x\overline{(\mathbf{S}_n\cdot\boldsymbol{\sigma})}\sigma_x \tag{79}$$

for an easy plane and

$$\sigma_x(\mathbf{S}_n\cdot\boldsymbol{\sigma})\sigma_x=-\sigma_z\overline{(\mathbf{S}_n\cdot\boldsymbol{\sigma})}\sigma_z \tag{80}$$

for an easy axis. When $n=1$, in terms of Eqs. (55), (56), (79), and (80), we can obtain

$$\mathbf{S}_1^x-i\mathbf{S}_1^y=[D_1(1)]_{12}\overline{[D_1(1)]_{21}}+[D_1(1)]_{11}\overline{[D_1(1)]_{22}} \tag{81}$$

and

$$\mathbf{S}_1^z=[D_1(1)]_{12}\overline{[D_1(1)]_{11}}+[D_1(1)]_{11}\overline{[D_1(1)]_{12}}, \tag{82}$$

where $D_1(1)$ can be rewritten as

$$D_1(1)=\frac{1}{(1-\xi_1)(1+\overline{\xi_1})\Delta_1^{1/2}}\begin{pmatrix}(\overline{\xi_1}|\delta_1|^2+\xi_1|\gamma_1|^2)(\overline{\xi_1}|\gamma_1|^2+\xi_1|\delta_1|^2) & 0\\ 0 & (\overline{\xi_1}|\gamma_1|^2+\xi_1|\delta_1|^2)(\overline{\xi_1}|\delta_1|^2+\xi_1|\gamma_1|^2)\end{pmatrix}$$
$$\times\begin{pmatrix}(1-\xi_1^2)\overline{\xi_1}|\delta_1|^2+(1-\overline{\xi_1^2})\xi_1|\gamma_1|^2 & (\xi_1^2-\overline{\xi_1^2})\overline{\gamma_1}\delta_1\\ (\xi_1^2-\overline{\xi_1^2})\overline{\delta_1}\gamma_1 & (1-\xi_1^2)\overline{\xi_1}|\gamma_1|^2+(1-\overline{\xi_1^2})\xi_1|\delta_1|^2\end{pmatrix}. \tag{83}$$

According to Eq. (74), only the relative values of $(b_1 1)$ have meaning, so one can find

$$(\gamma_1\delta_1)\sim(f_1 f_1^{-1})\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}, \tag{84}$$

where

$$f_1=\begin{cases} b_1^{1/2}\exp\left[i(\xi_1+\rho^2\xi_1^{-1})\left(x-2\dfrac{(\xi_1^2-\rho^2)^2}{\xi_1(\xi_1^2+\rho^2)}t\right)\right] & \text{for an easy plane}\\ b_1^{1/2}\exp\left[i(\xi_1-\rho^2\xi_1^{-1})\left(x-2\dfrac{(\xi_1^2+\rho^2)^2}{\xi_1(\xi_1^2-\rho^2)}t\right)\right] & \text{for an easy axis.}\end{cases} \tag{85}$$

Therefore,

$$\gamma_1=f_1+if_1^{-1},\quad \delta_1=f_1-if_1^{-1}, \tag{86}$$

while

$$f_1=\exp(-\Phi_1+i\Phi_2), \tag{87}$$

where

$$\Phi_1=\frac{2\xi_1''(|\xi_1|^2+\rho^2)}{|\xi_1|^2}(x-V_1t-x_{10}),\quad \Phi_2=\frac{2\xi_1'(|\xi_1|^2-\rho^2)}{|\xi_1|^2}(x-V_2t-x_{20}),$$

$$V_1=\frac{2\xi_1'(|\xi_1|^4+\rho^4)}{|\xi_1|^2(|\xi_1|^2+\rho^2)},\quad V_2=\frac{(\xi_1'^2-\xi_1''^2)(|\xi_1|^2+\rho^2)}{\xi_1'|\xi_1|^2},\tag{88}$$

for an easy plane and

$$\Phi_1=\frac{2\xi_1''(|\xi_1|^2+\rho^2)}{|\xi_1|^2}(x-V_1t-x_{10}),\quad \Phi_2=\frac{2\xi_1'(|\xi_1|^2-\rho^2)}{|\xi_1|^2}(x-V_2t-x_{20}),$$

$$V_1=\frac{4\xi_1'(|\xi_1|^2-\rho^2)}{|\xi_1|^2},\quad V_2=\frac{2[2\rho^2|\xi_1|^4+(\xi_1'^2-\xi_1''^2)(|\xi_1|^4+\rho^4)]}{\xi_1'|\xi_1|^2(|\xi_1|^2-\rho^2)}.\tag{89}$$

for an easy axis. By means of Eqs. (81)–(89), the single- soliton solutions can be written as

$$\mathbf{S}_1^x=S_0\sin\theta_0-\frac{2\xi_1''[\xi_1'(|\xi_1|^2-\rho^2)^2\cosh\Phi_1\cos\Phi_2+\xi_1''(|\xi_1|^4-\rho^4)\sinh\Phi_1\sin\Phi_2]}{|\xi_1|^2[(|\xi_1|^2-\rho^2)^2\cosh^2\Phi_1+4\rho^2\xi_1''^2\sin^2\Phi_2]},\tag{90}$$

$$\mathbf{S}_1^y=\frac{2\xi_1''[\xi_1''(|\xi_1|^2-\rho^2)^2\sinh\Phi_1\cos\Phi_2-\xi_1'(|\xi_1|^4-\rho^4)\cosh\Phi_1\sin\Phi_2]}{|\xi_1|^2[(|\xi_1|^2-\rho^2)^2\cosh^2\Phi_1+4\rho^2\xi_1''^2\sin^2\Phi_2]},\tag{91}$$

$$\mathbf{S}_1^z=S_0\cos\theta_0-\frac{2\xi_1''^2[(|\xi_1|^2-\rho^2)^2+4\rho^2|\xi_1|^2\sin^2\Phi_2]}{|\xi_1|^2[(|\xi_1|^2-\rho^2)^2\cosh^2\Phi_1+4\rho^2\xi_1''^2\sin^2\Phi_2]}\tag{92}$$

for an easy plane and

$$\mathbf{S}_1^x=\frac{2\xi_1''|\xi_1|(|\xi_1|^2+\rho^2)\cosh\Phi_1\cos\Phi_2}{[|\xi_1|^4+\rho^4-2\rho^2(\xi_1'^2-\xi_1''^2)]^{1/2}\{\xi_1''^2(|\xi_1|^2+\rho^2)^2[|\xi_1|^4+\rho^4-2\rho^2(\xi_1'^2-\xi_1''^2)]+|\xi_1|^2\cosh\Phi_1\}},\tag{93}$$

$$\mathbf{S}_1^y=\frac{2\xi_1''|\xi_1|(|\xi_1|^2+\rho^2)\cosh\Phi_1\sin\Phi_2}{[|\xi_1|^4+\rho^4-2\rho^2(\xi_1'^2-\xi_1''^2)]^{1/2}\{\xi_1''^2(|\xi_1|^2+\rho^2)^2[|\xi_1|^4+\rho^4-2\rho^2(\xi_1'^2-\xi_1''^2)]+|\xi_1|^2\cosh\Phi_1\}},\tag{94}$$

$$\mathbf{S}_1^z=S_0-\frac{2\xi_1''(|\xi_1|^2+\rho^2)^2}{\xi_1''^2(|\xi_1|^2+\rho^2)^2+|\xi_1|^2[|\xi_1|^4+\rho^4-2\rho^2(\xi_1'^2-\xi_1''^2)]\cosh^2\Phi_1},\tag{95}$$

for an easy axis. Similarly, we can also obtain the two-, three-, and multisoliton solutions.

These results show that the soliton solutions depend essentially on two velocities—V_1 in Eq. (88) and V_2 in Eq. (89)—which describe a spin configuration deviated from a homogeneous magnetization. The center of an inhomogeneity moves with a constant velocity V_1, while the shape of the soliton (the direction of a magnetization in its center) also changes with another velocity V_2.

In the polar coordinates, taking the z axis as the polar axis,

$$\cos\theta=\cos\theta_0-\frac{2\xi_1''^2[(|\xi_1|^2-\rho^2)^2+4\rho^2|\xi_1|^2\sin^2\Phi_2]}{|\xi_1|^2[(|\xi_1|^2-\rho^2)^2\cosh^2\Phi_1+4\rho^2\xi_1''^2\sin^2\Phi_2]},\tag{96}$$

for an easy plane and

$$\cos\theta=1-\frac{2\xi_1''(|\xi_1|^2+\rho^2)^2}{\xi_1''^2(|\xi_1|^2+\rho^2)^2+|\xi_1|^2[|\xi_1|^4+\rho^4-2\rho^2(\xi_1'^2-\xi_1''^2)]\cosh^2\Phi_1}\tag{97}$$

for an easy axis. We can find the property

$$\cos(-x,-t)=\cos(x,t).\tag{98}$$

In order to analyze the feature of the previous soliton solutions, setting the preliminary values as zero in the moving coordinates of the soliton,

$$\cos\theta=\cos\theta_0-\frac{2\xi_1''^2\left\{(|\xi_1|^2-\rho^2)^2+4\rho^2|\xi_1|^2\sin^2\left[\frac{2\xi_1'(|\xi_1|^2-\rho^2)}{|\xi_1|^2}(x-V_2t)\right]\right\}}{|\xi_1|^2\left\{(|\xi_1|^2-\rho^2)^2\cosh^2\left[\frac{2\xi_1''(|\xi_1|^2+\rho^2)}{|\xi_1|^2}x\right]+4\rho^2\xi_1''^2\sin^2\left[\frac{2\xi_1'(|\xi_1|^2-\rho^2)}{|\xi_1|^2}(x-V_2t)\right]\right\}} \tag{99}$$

for an easy plane and

$$\cos\theta=1-\frac{2\xi_1''(|\xi_1|^2+\rho^2)^2}{\xi_1''^2(|\xi_1|^2+\rho^2)^2+|\xi_1|^2[|\xi_1|^4+\rho^4-2\rho^2(\xi_1'^2-\xi_1''^2)]\cosh^2\left[\frac{2\xi_1''(|\xi_1|^2+\rho^2)}{|\xi_1|^2}x\right]} \tag{100}$$

for an easy axis. Therefore, the depths and widths of the surface of solitary waves are not constants, but vary periodically with time. The shape of solitary waves in a spin chain with an easy plane also changes with a velocity V_2 and it is not symmetrical with respect to the center, while the shape of solitary waves in a spin chain with an easy axis is symmetrical with respect to the center. In a spin chain with an easy plane, the integral of the motion coincident with the z component of the total magnetic momentum

$$P^z=S_0\int dx(1-\cos\theta) \tag{101}$$

is not a constant and is dependent on time periodically, while P^z in a spin chain with an easy axis is a constant, where P^z has the sense of the mean number of spin deviated from the ground state in a localized magnetic excitation. This feature did not appear in the soliton solution of all other nonlinear equations solved.

While

$$\tan\varphi=\frac{\xi_1''(|\xi_1|^2-\rho^2)^2\sinh\Phi_1\cos\Phi_2-\xi_1'(|\xi_1|^4-\rho^4)\cosh\Phi_1\sin\Phi_2}{\xi_1'(|\xi_1|^2-\rho^2)^2\cosh\Phi_1\cos\Phi_2+\xi_1''(|\xi_1|^4-\rho^4)\sinh\Phi_1\sin\Phi_2} \tag{102}$$

for an easy plane and

$$\tan\varphi=\frac{[\xi_1''(|\xi_1|^2-\rho^2)^2+4\rho^2\xi_1'|\xi_1|^2]\sinh\Phi_1\cos\Phi_2-\xi_1'(|\xi_1|^4-\rho^4)\cosh\Phi_1\sin\Phi_2}{\xi_1'(|\xi_1|^4-\rho^4)\cosh\Phi_1\cos\Phi_2+[\xi_1''(|\xi_1|^2-\rho^2)^2+4\rho^2\xi_1'|\xi_1|^2]\sinh\Phi_1\sin\Phi_2} \tag{103}$$

for an easy axis, setting the preliminary values as zero in the moving coordinates of the soliton,

$$\tan\varphi=\frac{\sinh\left[\frac{2\xi_1''(|\xi_1|^2+\rho^2)}{|\xi_1|^2}x\right]-\frac{\xi_1'(|\xi_1|^2+\rho^2)}{\xi_1''(|\xi_1|^2-\rho^2)}\cosh\left[\frac{2\xi_1''(|\xi_1|^2+\rho^2)}{|\xi_1|^2}x\right]\tan\left[\frac{2\xi_1'(|\xi_1|^2-\rho^2)}{|\xi_1|^2}(x-V_2t)\right]}{\frac{\xi_1'(|\xi_1|^2-\rho^2)}{\xi_1''(|\xi_1|^2+\rho^2)}\cosh\left[\frac{2\xi_1''(|\xi_1|^2+\rho^2)}{|\xi_1|^2}x\right]+\sinh\left[\frac{2\xi_1''(|\xi_1|^2+\rho^2)}{|\xi_1|^2}x\right]\tan\left[\frac{2\xi_1'(|\xi_1|^2-\rho^2)}{|\xi_1|^2}(x-V_2t)\right]} \tag{104}$$

for an easy plane and

$$\tan\varphi=\frac{\sinh\left[\frac{2\xi_1''(|\xi_1|^2+\rho^2)}{|\xi_1|^2}x\right]-\frac{\xi_1'(|\xi_1|^4-\rho^4)}{\xi_1''(|\xi_1|^2-\rho^2)^2+4\rho^2\xi_1'|\xi_1|^2}\cosh\left[\frac{2\xi_1''(|\xi_1|^2+\rho^2)}{|\xi_1|^2}x\right]\tan\left[\frac{2\xi_1'(|\xi_1|^2-\rho^2)}{|\xi_1|^2}(x-V_2t)\right]}{\frac{\xi_1'(|\xi_1|^4-\rho^4)}{\xi_1''(|\xi_1|^2-\rho^2)^2+4\rho^2\xi_1'|\xi_1|^2}\cosh\left[\frac{2\xi_1''(|\xi_1|^2+\rho^2)}{|\xi_1|^2}x\right]+\sinh\left[\frac{2\xi_1''(|\xi_1|^2+\rho^2)}{|\xi_1|^2}x\right]\tan\left[\frac{2\xi_1'(|\xi_1|^2-\rho^2)}{|\xi_1|^2}(x-V_2t)\right]} \tag{105}$$

for an easy axis. The total magnetic momentum

$$\mathbf{P}=S_0\int dx(1-\cos\theta)\nabla\varphi \tag{106}$$

also is not constant. These properties are important for the classical Heisenberg spin chain with a uniaxial anisotropy in an external magnetic field, but they have never been obtained by all other methods.

Obviously, when an anisotropic parameter $\rho\to 0$, these soliton solutions in the classical Heisenberg spin chain with a uniaxial anisotropy reduce to those in an isotropic spin chain, for example, the single-soliton solutions (90)–(95) are transformed to

$$\begin{aligned}\mathbf{S}_1^x=&\frac{2\xi_1''}{|\xi_1|^2}\mathrm{sech}^2[\xi_1''(x-4\xi_1't-x_{10})]\\&\times\Bigg(\xi_1''\sinh[\xi_1''(x-4\xi_1't-x_{10})]\sin\\&\times\left\{\xi_1'\left[x-2\left(\xi_1'-\frac{\xi_1''^2}{\xi_1'}\right)t-x_{20}\right]\right\}\\&+\xi_1'\cosh[\xi_1''(x-4\xi_1't-x_{10})]\cos\\&\times\left\{\xi_1'\left[x-2\left(\xi_1'-\frac{\xi_1''^2}{\xi_1'}\right)t-x_{20}\right]\right\}\Bigg),\end{aligned} \tag{107}$$

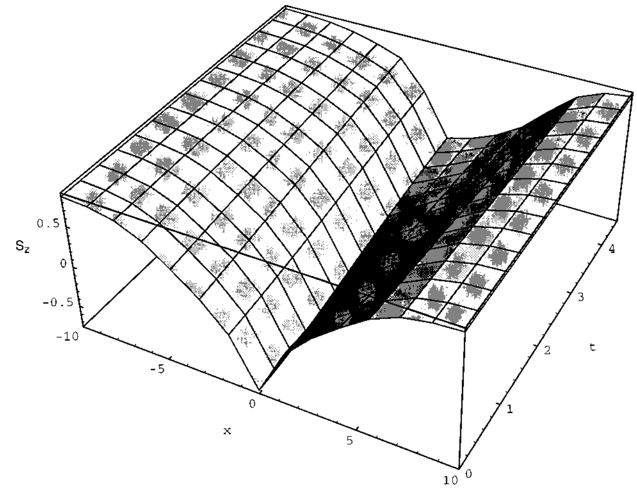
$$\begin{aligned}\mathbf{S}_1^y=&\frac{2\xi_1''}{|\xi_1|^2}\mathrm{sech}^2[\xi_1''(x-4\xi_1't-x_{10})]\\&\times\Bigg(\xi_1''\sinh[\xi_1''(x-4\xi_1't-x_{10})]\\&\times\cos\left\{\xi_1'\left[x-2\left(\xi_1'-\frac{\xi_1''^2}{\xi_1'}\right)t-x_{20}\right]\right\}\\&-\xi_1'\cosh[\xi_1''(x-4\xi_1't-x_{10})]\\&\times\sin\left\{\xi_1'\left[x-2\left(\xi_1'-\frac{\xi_1''^2}{\xi_1'}\right)t-x_{20}\right]\right\}\Bigg),\end{aligned} \tag{108}$$

$$\mathbf{S}_1^z=S_0-\frac{2\xi_1''^2}{|\xi_1|^2}\mathrm{sech}^2[\xi_1''(x-4\xi_1't-x_{10})]. \tag{109}$$

These results are equal to Eq. (27a) obtained by the method of an inverse scattering transformation in Ref. [40]. While taking the z axis as the polar axis in the polar coordinates,

$$\cos\theta=1-\frac{2\xi_1''^2}{|\xi_1|^2}\mathrm{sech}^2[\xi_1''(x-4\xi_1't-x_{10})], \tag{110}$$

$$\begin{aligned}\varphi=&\varphi_0+\xi_1'\left[x-2\left(\xi_1'-\frac{\xi_1''^2}{\xi_1'}\right)t-x_{20}\right]\\&+\tan^{-1}\left\{\frac{\xi_1''}{\xi_1'}\tanh[\xi_1''(x-4\xi_1't-x_{10})]\right\},\end{aligned} \tag{111}$$



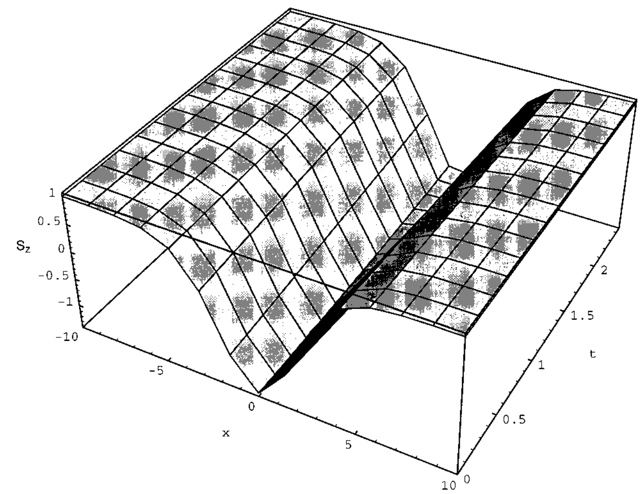


FIG. 1. (a) Graphical illustrations of the motion of the center and the change of shape of a soliton solution $\mathbf{S}_1^z$ expressed by Eq. (92) in the classical Heisenberg spin chain with an easy plane, where $\rho=0.1$, $\xi_1'=0.1$, $\xi_1''=0.2$, $x_{10}=0$, $x_{20}=0$, and $\pi/4V_1$ as a unit of time. (b) Graphical illustrations of the motion of the center and the change of shape of a soliton solution $\mathbf{S}_1^z$ expressed by Eq. (95) in the classical Heisenberg spin chain with an easy axis, where $\rho=0.1$, $\xi_1'=0.1$, $\xi_1''=0.2$, $x_{10}=0$, $x_{20}=0$, and $\pi/4V_1$ as a unit of time.

when $t\to 0$, these results are equivalent to Eq. (22) obtained by means of the method of separating variables in the moving coordinates in Ref. [19].

In terms of soliton solutions (110) and (111) in an isotropic spin chain, we can find that the z components of the total magnetic moment P_z and the total magnetic momentum $\mathbf{P}$ are constants of motion, $P^z=4S_0\xi_1''/(|\xi_1|^2)$ and $P=4S_0\sin^{-1}(\xi_1''/|\xi_1|)$. Tjon and Wright [19] took advantage of this feature in solving the equation of motion. These properties are important for the classical Heisenberg spin chain with a uniaxial anisotropy in an external magnetic field, but they have never been obtained by all other methods.

Figures 1–4 give some graphical illustrations of the motion of the center and the change of shape of a previous soliton solution $\mathbf{S}_1^z$ expressed by Eqs. (92) and (95) in an anisotropic spin chain and that by Eq. (109) in an isotropic

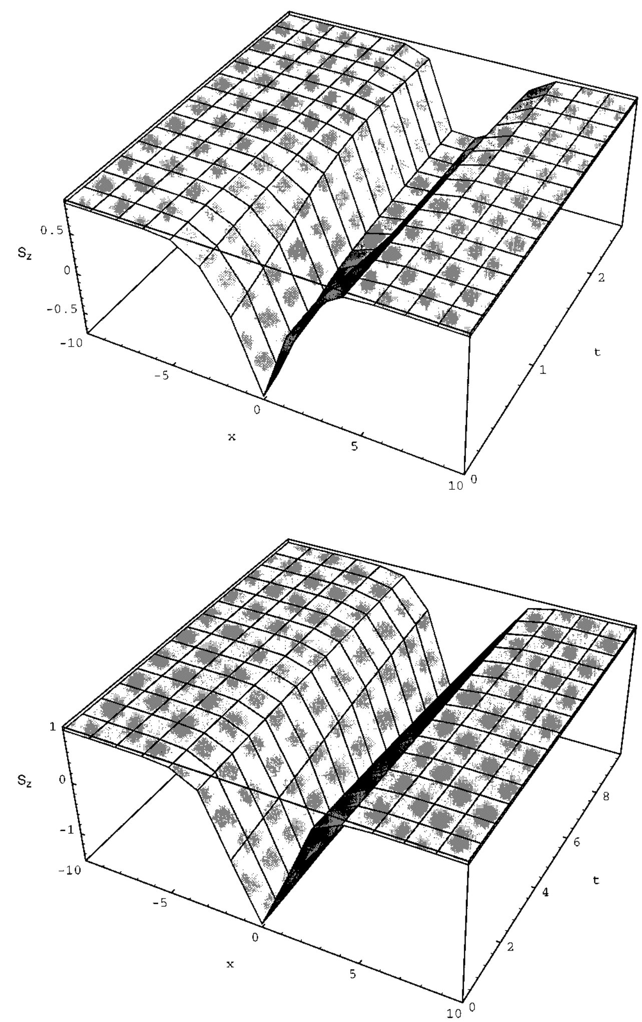


FIG. 2. (a) Graphical illustrations of the motion of the center and the change of shape of a soliton solution $\mathbf{S}_1^z$ expressed by Eq. (92) in the classical Heisenberg spin chain with an easy plane, where $\rho=0.3$, $\xi_1'=0.1$, $\xi_1''=0.2$, $x_{10}=0$, $x_{20}=0$, and $\pi/4V_1$ as a unit of time. (b) Graphical illustrations of the motion of the center and the change of shape of a soliton solution $\mathbf{S}_1^z$ expressed by Eq. (95) in the classical Heisenberg spin chain with an easy axis, where $\rho=0.3$, $\xi_1'=0.1$, $\xi_1''=0.2$, $x_{10}=0$, $x_{20}=0$, and $\pi/4V_1$ as a unit of time.

spin chain. In the figures, we took the parameters $\xi_1'=0.1$, $\xi_1''=0.2$, $x_{10}=0$, $x_{20}=0$, and $\pi/4V_1$ as a unit of time and then set $\rho=0.10$ in Figs. 1 and 4, $\rho=0.3$ in Fig. 2, and $\rho=0$ in Fig. 3, respectively. If the x-$\mathbf{S}_1^z$ plane is taken a reference plane when $t=0$, we can directly find the following feature of solitary wave $\mathbf{S}_1^z$.

(i) Since the lowest point of the surface is located in the plane of the center of the surface, we can observe the motion of the center by looking at the motion of the lowest point. The lowest point of the surface in the figures moves with five constant velocities V_1 corresponding to anisotropic parameters ρ, respectively.

(ii) The shape of the surface of $\mathbf{S}_1^z$ changes with velocity V_2 and the surface is not symmetrical with respect to the center, as illustrated by Figs. 1 and 2, respectively. When $\rho\rightarrow 0$, the soliton solution $\mathbf{S}_1^z$, expressed by Eqs. (92) and (95) in an anisotropic spin chain, reduces to that in Eq. (109) in an isotropic spin chain. The shape of the surface of $\mathbf{S}_1^z$ does not change with velocity V_2 and the surface is symmetrical with respect to the center, as shown in Fig. 3.

(iii) The depth and width of the surface of $\mathbf{S}_1^z$ are not constants but vary periodically with time, as shown in Fig. 4. When $\rho\rightarrow 0$, the depth and width of the surface of $\mathbf{S}_1^z$, expressed by Eq. (109) in an isotropic spin chain, do not change periodically with time and the surface is also symmetrical with respect to the center, as illustrated by Fig. 3.

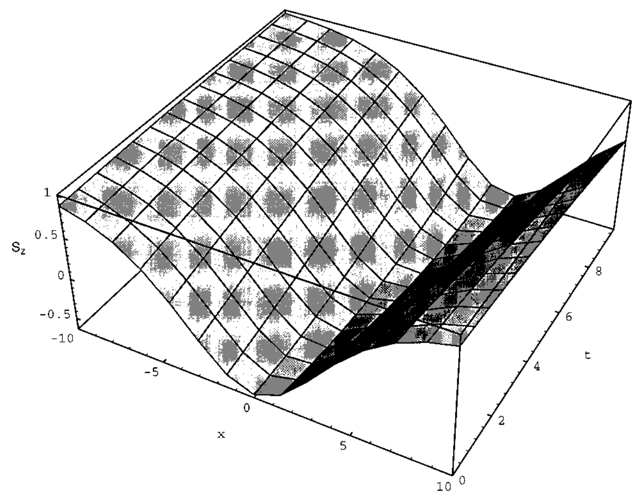


FIG. 3. Graphical illustrations of a soliton solution $\mathbf{S}_1^z$ expressed by Eq. (109) in an isotropic Heisenberg spin chain, where $\rho=0$, $\xi_1'=0.1$, $\xi_1''=0.2$, $x_{10}=0$, $x_{20}=0$, and $\pi/4V_1$ as a unit of time.

IV. ASYMPTOTIC BEHAVIOR OF MULTISOLITON SOLUTIONS

In this section we will construct a direct procedure for studying the asymptotic behavior of multisoliton solutions in

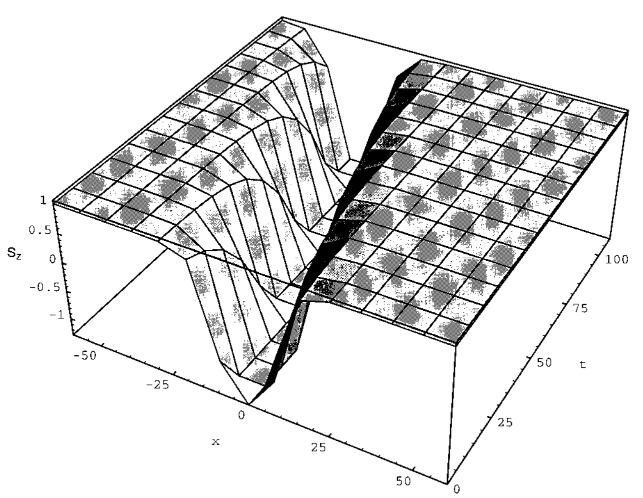


FIG. 4. Graphical illustrations of the depth and width of the surface of a soliton solution $\mathbf{S}_1^z$ expressed by Eq. (92) in the classical Heisenberg spin chain with an easy plane changing periodically with time, where $\rho=0.1$, $\xi_1'=0.1$, $\xi_1''=0.2$, $x_{10}=0$, $x_{20}=0$, and $\pi/4V_1$ as a unit of time.

the classical Heisenberg spin chain with uniaxial anisotropy in an external magnetic field. According to Eq. (42), we can define

$$\Psi_N(\xi)=J_N(\xi)\Psi_0(\xi), \tag{112}$$

while

$$J_N(\xi)=D_N(\xi)D_{N-1}(\xi)\cdots D_1(\xi), \tag{113}$$

where $J_N(\xi)$ has N pairs of poles ξ_n and $-\bar{\xi}_n$, $n=1,2,\ldots,N$. Similar to Eq. (28), we can obtain the Lax equations for $\Psi_N(\xi)$,

$$\partial_x\Psi_N(\xi)=L_N(\xi)\Psi_N(\xi),$$

$$\partial_t\Psi_N(\xi)=M_N(\xi)\Psi_N(\xi). \tag{114}$$

On the basis of Eq. (48), $J_N(\xi)$ can be written as

$$J_N(\xi)=K_N P_N(\xi), \tag{115}$$

while

$$K_N(\xi)=C_N C_{N-1}\cdots C_1 \tag{116}$$

and

$$P_N(\xi)=I-\sum_{n=1}^{N}\frac{1}{\xi_n-\xi}G_n+\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n+\xi}\widetilde{G}_n, \tag{117}$$

where K_N is a 2×2 matrix independent of ξ, i.e.,

$$K_N(\xi)=\begin{cases}\exp\left(\dfrac{i}{2}\Theta_N(\xi)\sigma_z\right) & \text{for an easy plane}\\ \exp\left(\dfrac{i}{2}\Theta_N(\xi)\sigma_z\right) & \text{for an easy axis,}\end{cases} \tag{118}$$

with

$$\Theta_N(\xi)=\sum_{n=1}^{N}\phi_n(\xi). \tag{119}$$

By means of Eqs. (51) and (52), we can obtain the relations

$$J_N(\xi)=\begin{cases}\sigma_x\overline{J(-\bar{\xi})}\sigma_x & \text{for an easy plane}\\ \sigma_z\overline{J(-\bar{\xi})}\sigma_z & \text{for an easy axis}\end{cases} \tag{120}$$

and

$$J_N^{\dagger}(\bar{\xi})=J_N^{-1}(\xi),\quad J_N(\xi)J_N^{-1}(\xi)=J_N^{-1}(\xi)J_N(\xi)=I, \tag{121}$$

while

$$P_N^{\dagger}(\bar{\xi})=\begin{cases}I-\displaystyle\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n-\xi}G_n^{\dagger}-\sum_{n=1}^{N}\frac{1}{\xi_n+\xi}\sigma_x G_n^T\sigma_x & \text{for an easy plane}\\ I-\displaystyle\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n-\xi}G_n^{\dagger}-\sum_{n=1}^{N}\frac{1}{\xi_n+\xi}\sigma_z G_n^T\sigma_z & \text{for an easy axis}\end{cases} \tag{122}$$

and

$$P_N^{-1}(\xi)=P_N^{\dagger}(\bar{\xi}), \tag{123}$$

where

$$\widetilde{G}_n=\begin{cases}-\sigma_x\overline{G_n}\sigma_x & \text{for an easy plane}\\ -\sigma_z\overline{G_n}\sigma_z & \text{for an easy axis.}\end{cases} \tag{124}$$

Because $J_N(\xi)J_N^{-1}(\xi)=J_N^{-1}(\xi)J_N(\xi)=I$ in Eq. (121), its residue at $\xi=\xi_n$ should vanish, i.e., $G_m P_N^{\dagger}(\bar{\xi}_m)=0$,

$$G_m\left(I-\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n-\xi_m}G_n^{\dagger}-\sum_{n=1}^{N}\frac{1}{\xi_n+\xi_m}\sigma_x G_n^T\sigma_x\right)=0 \tag{125}$$

for an easy plane and

$$G_m\left(I-\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n-\xi_m}G_n^{\dagger}-\sum_{n=1}^{N}\frac{1}{\xi_n+\xi_m}\sigma_z G_n^T\sigma_z\right)=0 \tag{126}$$

for an easy axis. This result shows that G_m is degenerate; it can be defined

$$G_n=(\alpha_n'\beta_n')^T(\gamma_n'\delta_n'). \tag{127}$$

In order to solve Eqs. (125) and (126), we can introduce a transformation

$$J_N'(\xi)=U^{-1}J_N(\xi)U, \tag{128}$$

and

$$G_n'=U^{-1}G_nU,\quad \widetilde{G}_n'=U^{-1}\widetilde{G}_nU=-\overline{G_n'}, \tag{129}$$

where $U^{-1}\sigma_x\bar{U}=i$. Corresponding to Eqs. (125) and (126), we can write

$$G'_m\left(I-\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n-\xi_m}G'^{\dagger}_n-\sum_{n=1}^{N}\frac{1}{\xi_n+\xi_m}(G'_n)^T\right)=0. \tag{130}$$

Taking the limit $\xi\to\xi_n$ in Eq. (114), we obtain

$$\partial_x[K_N G_n\Psi_0(\xi_n)]=L_N(\xi_n)[K_N G_n\Psi_0(\xi_n)],$$

$$\partial_t[K_N G_n\Psi_0(\xi_n)]=M_N(\xi_n)[K_N G_n\Psi_0(\xi_n)]. \tag{131}$$

Because G_n is degenerate, the factor

$$(\gamma'_n\delta'_n)\Psi_0(\xi_n) \tag{132}$$

must be independent of x and t. Therefore, we can simply obtain

$$(\gamma'_n\delta'_n)=(b_n 1)\Psi_0^{-1}(\xi_n), \tag{133}$$

where b_n is a constant that has been shown in Eq. (84), while α'_n, β'_n, γ'_n, and δ'_n are different from α_n, β_n, γ_n, and δ_n, except that $\gamma'_1=\gamma_1$ and $\delta'_1=\delta_1$. Noting Eq. (132), G'_n can be expressed as

$$G'_n=(\rho_n\nu_n)^T(f_n f_n^{-1}), \tag{134}$$

where

$$f_1=\begin{cases} b_1^{1/2}\exp\left[i(\xi_n+\rho^2\xi_1^{-1})\left(x-2\dfrac{(\xi_n^2-\rho^2)^2}{\xi_n(\xi_n^2+\rho^2)}t\right)\right] & \text{for an easy plane}\\ b_1^{1/2}\exp\left[i(\xi_n-\rho^2\xi_n^{-1})\left(x-2\dfrac{(\xi_n^2+\rho^2)^2}{\xi_n(\xi_n^2-\rho^2)}t\right)\right] & \text{for an easy axis.}\end{cases} \tag{135}$$

Substituting Eq. (134) into Eq. (130), we obtain

$$f_m=\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n-\xi_m}(f_m\bar{f}_n+f_m^{-1}\bar{f}_n^{-1})\bar{\rho}_n+\sum_{n=1}^{N}\frac{1}{\xi_n+\xi_m}(f_m f_n+f_m^{-1}f_n^{-1})\rho_n \tag{136}$$

and

$$f_m^{-1}=\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n-\xi_m}(f_m\bar{f}_n+f_m^{-1}\bar{f}_n^{-1})\bar{\nu}_n+\sum_{n=1}^{N}\frac{1}{\xi_n+\xi_m}(f_m f_n+f_m^{-1}f_n^{-1})\nu_n. \tag{137}$$

By means of Eqs. (136) and (137), one can find ρ_n,ν_n, and $P'_N(\xi)$, e.g.,

$$P'_N(1)_{11}=1-\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n+1}\bar{\rho}_n\bar{f}_n-\sum_{n=1}^{N}\frac{1}{\xi_n-1}\rho_n f_n. \tag{138}$$

According to Eqs. (136) and (137), we obtain

$$1=\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n-\xi_m}(1+f_m^{-2}\bar{f}_n^{-2})\overline{\rho_n f_n}+\sum_{n=1}^{N}\frac{1}{\xi_n+\xi_m}(1+f_m^{-2}f_n^{-2})\rho_n f_n \tag{139}$$

and

$$1=\sum_{n=1}^{N}\frac{1}{\bar{\xi}_n+\bar{\xi}_m}(1+\bar{f}_m^{-2}\bar{f}_n^{-2})\overline{\rho_n f_n}+\sum_{n=1}^{N}\frac{1}{\xi_n-\bar{\xi}_m}(1+\bar{f}_m^{-2}f_n^{-2})\rho_n f_n. \tag{140}$$

In terms of Eqs. (139) and (140), $\rho_n,\bar{\rho}_n,P'_N(\xi)_{11}$, and $P'_N(\xi)_{12}$ can be easily determined. However, although ρ_n and $\bar{\rho}_n$ appear in both Eqs. (139) and (140), it is hard to obtain explicit expressions of them by the well-known Binet-Cauchy formula. The asymptotic behaviors of the multisoliton solutions can be derived from them.

Introducing

$$\Delta_l=\begin{cases}\rho_n f_n & \text{if } n=l,\quad l\in 1,2,\ldots,N\\ \overline{\rho_n f_n} & \text{if } n=l-N,\quad l\in N+1,N+2,\ldots,2N\end{cases} \tag{141}$$

and

$$E_n=1,\quad l\in 1,2,\ldots,2N, \tag{142}$$

where E is a row matrix, Eqs. (139) and (140) can be expressed by

$$E=\Delta Q, \tag{143}$$

where Q is a $2N\times 2N$ matrix,

$$Q_{n,m}=\frac{1}{\xi_n+\xi_m}(1+f_n^{-2}f_m^{-2}), \tag{144}$$

$$Q_{n,N+m}=\frac{1}{\xi_n-\bar{\xi}_m}(1+f_n^{-2}\bar{f}_m^{-2}), \tag{145}$$

$$Q_{N+n,m}=\frac{1}{\bar{\xi}_n-\xi_m}(1+\bar{f}_n^{-2}f_m^{-2}), \quad (146)$$

$$Q_{N+n,N+m}=\frac{1}{\bar{\xi}_n+\bar{\xi}_m}(1+\bar{f}_n^{-2}\bar{f}_m^{-2}). \quad (147)$$

By means of Eq. (142),

$$\Delta=EQ^{-1}. \quad (148)$$

$P_N'(1)_{11}$ in Eq. (138) can be written as

$$P_N'(1)_{11}=1+\sum_{l=1}^{2N}\Delta_l R_l=1+\Delta R^T, \quad (149)$$

where

$$R_l=\begin{cases}\dfrac{-1}{\xi_n-1} & \text{if } n=l, \quad l\in 1,2,\ldots,N\\[2ex] \dfrac{-1}{\bar{\xi}_n+1} & \text{if } n=l-N, \quad l\in N+1,N+2,\ldots,2N.\end{cases} \quad (150)$$

According to Eq. (142), $P_N'(1)_{11}$ in Eq. (149) can be expressed as

$$P_N'(1)_{11}=1+EQ^{-1}R^T=\frac{\det(Q+R^TE)}{\det Q}. \quad (151)$$

When $N=1$ and $\xi=\xi_j$, $\det Q$ is written as

$$\det Q=\det\begin{pmatrix}\dfrac{1}{2\xi_j}(1+f_j^{-4}) & \dfrac{1}{\xi_j-\bar{\xi}_j}(1+|f_j|^{-4})\\[2ex] \dfrac{1}{\bar{\xi}_j-\xi_j}(1+|f_j|^{-4}) & \dfrac{1}{2\bar{\xi}_j}(1+\bar{f}_j^{-4})\end{pmatrix}. \quad (152)$$

By means of Eq. (135), f_n can be written as

$$f_n=\exp(-\Phi_{1n}+i\Phi_{2n}), \quad (153)$$

where

$$\Phi_{1n}=\frac{2\xi_n''(|\xi_n|^2+\rho^2)}{|\xi_n|^2}(x-V_{1n}t-x_{1n0}),$$

$$\Phi_{2n}=\frac{2\xi_n'(|\xi_n|^2-\rho^2)}{|\xi_n|^2}(x-V_{2n}t-x_{2n0}), \quad (154)$$

$$V_{1n}=\frac{2\xi_n'(|\xi_n|^4+\rho^4)}{|\xi_n|^2(|\xi_n|^2+\rho^2)},$$

$$V_{2n}=\frac{(\xi_n'^2-\xi_n''^2)(|\xi_n|^2+\rho^2)}{\xi_n'|\xi_n|^2}$$

for an easy plane and

$$\Phi_{1n}=\frac{2\xi_n''(|\xi_n|^2+\rho^2)}{|\xi_n|^2}(x-V_{1n}t-x_{1n0}),$$

$$\Phi_{2n}=\frac{2\xi_n'(|\xi_n|^2-\rho^2)}{|\xi_n|^2}(x-V_{2n}t-x_{2n0}), \quad (155)$$

$$V_{1n}=\frac{4\xi_n'(|\xi_n|^2-\rho^2)}{|\xi_n|^2},$$

$$V_{2n}=\frac{2[2\rho^2|\xi_n|^4+(\xi_n'^2-\xi_n''^2)(|\xi_n|^4+\rho^4)]}{\xi_n'|\xi_n|^2(|\xi_n|^2-\rho^2)}$$

for an easy axis.

Suppose all $\xi_n''>0$ and $V_{1N}>V_{1(N-1)}>\cdots>V_{11}$, and the vicinity of $V_{1n}t-x_{1n0}$ is denoted by Θ_n. For extremely large t, these vicinities are separated from left to right as Θ_N, $\Theta_{N-1},\ldots,\Theta_1$. In the vicinity Θ_j, we have the limits

$$(x-V_{1n}t-x_{1n0})\to-\infty, \quad |f_n|^{-1}\to 0 \quad \text{if } n<j,$$

$$(x-V_{1n}t-x_{1n0})\to\infty, \quad |f_n|^{-1}\to\infty \quad \text{if } m>j, \quad (156)$$

while $\det Q$ tends to

$$\begin{vmatrix} \frac{1}{\xi_n+\xi_{n'}} & \frac{1}{\xi_n+\xi_j} & 0 & \frac{1}{\xi_n-\bar{\xi}_{n'}} & \frac{1}{\xi_n-\bar{\xi}_j} & 0 \\ \frac{1}{\xi_j+\xi_{n'}} & \frac{1+f_j^{-4}}{2\xi_j} & \frac{f_j^{-2}f_{m'}^{-2}}{\xi_j+\xi_{m'}} & \frac{1}{\xi_j-\bar{\xi}_{n'}} & \frac{1+|f_j|^{-4}}{\xi_j-\bar{\xi}_j} & \frac{f_j^{-2}\bar{f}_{m'}^{-2}}{\xi_j-\bar{\xi}_{m'}} \\ 0 & \frac{f_m^{-2}f_j^{-2}}{\xi_m+\xi_j} & \frac{f_m^{-2}f_{m'}^{-2}}{\xi_m+\xi_{m'}} & 0 & \frac{f_m^{-2}\bar{f}_j^{-2}}{\xi_m-\bar{\xi}_j} & \frac{f_m^{-2}\bar{f}_{m'}^{-2}}{\xi_m-\bar{\xi}_{m'}} \\ \frac{1}{\bar{\xi}_n-\xi_{n'}} & \frac{1}{\bar{\xi}_n-\xi_j} & 0 & \frac{1}{\bar{\xi}_n+\bar{\xi}_{n'}} & \frac{1}{\bar{\xi}_n+\bar{\xi}_j} & 0 \\ \frac{1}{\bar{\xi}_j-\xi_{n'}} & \frac{1+|f_j|^{-4}}{\bar{\xi}_j-\xi_j} & \frac{\bar{f}_j^{-2}f_{m'}^{-2}}{\bar{\xi}_j-\xi_{m'}} & \frac{1}{\bar{\xi}_j+\bar{\xi}_{n'}} & \frac{1+\bar{f}_j^{-4}}{2\bar{\xi}_j} & \frac{\bar{f}_j^{-2}\bar{f}_{m'}^{-2}}{\bar{\xi}_j+\bar{\xi}_{m'}} \\ 0 & \frac{\bar{f}_m^{-2}f_j^{-2}}{\bar{\xi}_m-\xi_j} & \frac{\bar{f}_m^{-2}f_{m'}^{-2}}{\bar{\xi}_m-\xi_{m'}} & 0 & \frac{\bar{f}_m^{-2}\bar{f}_j^{-2}}{\bar{\xi}_m+\bar{\xi}_j} & \frac{\bar{f}_m^{-2}\bar{f}_{m'}^{-2}}{\bar{\xi}_m+\bar{\xi}_{m'}} \end{vmatrix}, \tag{157}$$

where $n,n'<j<m,m'$.

Now only those terms leading to $|f_{j+1}|^{-8}\cdots|f_N|^{-8}$ remain. It is hard to calculate this determinant. Similar to the procedure in Ref. [38], we consider the term without f_j,

$$\begin{vmatrix} \frac{1}{\xi_n+\xi_{n'}} & \frac{1}{\xi_n+\xi_j} & 0 & \frac{1}{\xi_n-\bar{\xi}_{n'}} & \frac{1}{\xi_n-\bar{\xi}_j} & 0 \\ \frac{1}{\xi_j+\xi_{n'}} & \frac{1}{2\xi_j} & 0 & \frac{1}{\xi_j-\bar{\xi}_{n'}} & \frac{1}{\xi_j-\bar{\xi}_j} & 0 \\ 0 & 0 & \frac{f_m^{-2}f_{m'}^{-2}}{\xi_m+\xi_{m'}} & 0 & 0 & \frac{f_m^{-2}\bar{f}_{m'}^{-2}}{\xi_m-\bar{\xi}_{m'}} \\ \frac{1}{\bar{\xi}_n-\xi_{n'}} & \frac{1}{\bar{\xi}_n-\xi_j} & 0 & \frac{1}{\bar{\xi}_n+\bar{\xi}_{n'}} & \frac{1}{\bar{\xi}_n+\bar{\xi}_j} & 0 \\ \frac{1}{\bar{\xi}_j-\xi_{n'}} & \frac{1}{\bar{\xi}_j-\xi_j} & 0 & \frac{1}{\bar{\xi}_j+\bar{\xi}_{n'}} & \frac{1}{2\bar{\xi}_j} & 0 \\ 0 & 0 & \frac{\bar{f}_m^{-2}f_{m'}^{-2}}{\bar{\xi}_m-\xi_{m'}} & 0 & 0 & \frac{\bar{f}_m^{-2}\bar{f}_{m'}^{-2}}{\bar{\xi}_m+\bar{\xi}_{m'}} \end{vmatrix}. \tag{158}$$

The term involving f_j^{-4} is the determinant

$$\left|\begin{array}{cccccc} \frac{1}{\xi_n+\xi_{n'}} & 0 & 0 & \frac{1}{\xi_n-\overline{\xi_{n'}}} & \frac{1}{\xi_n-\overline{\xi_j}} & 0 \\ 0 & \frac{f_j^{-4}}{2\xi_j} & \frac{f_j^{-2}f_{m'}^{-2}}{\xi_j+\xi_{m'}} & 0 & 0 & \frac{f_j^{-2}\overline{f_{m'}}^{-2}}{\xi_j-\overline{\xi_{m'}}} \\ 0 & \frac{f_m^{-2}f_j^{-2}}{\xi_m+\xi_j} & \frac{f_m^{-2}f_{m'}^{-2}}{\xi_m+\xi_{m'}} & 0 & 0 & \frac{f_m^{-2}\overline{f_{m'}}^{-2}}{\xi_m-\overline{\xi_{m'}}} \\ \frac{1}{\overline{\xi_n}-\xi_{n'}} & 0 & 0 & \frac{1}{\overline{\xi_n}+\overline{\xi_{n'}}} & \frac{1}{\overline{\xi_n}+\overline{\xi_j}} & 0 \\ \frac{1}{\overline{\xi_j}-\xi_{n'}} & 0 & 0 & \frac{1}{\overline{\xi_j}+\overline{\xi_{n'}}} & \frac{1}{2\overline{\xi_j}} & 0 \\ 0 & \frac{\overline{f_m}^{-2}f_j^{-2}}{\overline{\xi_m}-\xi_j} & \frac{\overline{f_m}^{-2}f_{m'}^{-2}}{\overline{\xi_m}-\xi_{m'}} & 0 & 0 & \frac{\overline{f_m}^{-2}\overline{f_{m'}}^{-2}}{\overline{\xi_m}+\overline{\xi_{m'}}} \end{array}\right|, \tag{159}$$

In addition to the common factor $|f_{j+1}|^{-8}\cdots|f_N|^{-8}$, these two determinants are clearly proportional to

$$\left|\begin{array}{ccc} \frac{1}{\xi_n+\xi_{n'}} & \frac{1}{\xi_n-\overline{\xi_{n'}}} & \frac{1}{\xi_n-\overline{\xi_j}} \\ \frac{1}{\overline{\xi_n}-\xi_{n'}} & \frac{1}{\overline{\xi_n}+\overline{\xi_{n'}}} & \frac{1}{\overline{\xi_n}+\overline{\xi_j}} \\ \frac{1}{\overline{\xi_n}-\xi_{n'}} & \frac{1}{\xi_n+\overline{\xi_{n'}}} & \frac{1}{2\overline{\xi_j}} \end{array}\right| \left|\begin{array}{cc} \frac{1}{\xi_m+\xi_{m'}} & \frac{1}{\xi_m-\overline{\xi_{m'}}} \\ \frac{1}{\overline{\xi_m}-\xi_{m'}} & \frac{1}{\overline{\xi_m}+\overline{\xi_{m'}}} \end{array}\right|. \tag{160}$$

The proportional coefficients are

$$\frac{(\xi_j+\overline{\xi_j})^2}{2\xi_j|\xi_j-\overline{\xi_j}|^2}\prod_{n=1}^{j-1}\frac{(\xi_j-\xi_n)^2(\xi_j+\overline{\xi_n})^2}{(\xi_j+\xi_n)^2(\xi_j-\overline{\xi_n})^2} \tag{161}$$

and

$$-\frac{1}{2\xi_j}\prod_{m=j+1}^{N}\frac{(\xi_j-\xi_m)^2(\xi_j+\overline{\xi_m})^2}{(\xi_j+\xi_m)^2(\xi_j-\overline{\xi_m})^2}. \tag{162}$$

Therefore, the asymptotic behavior of the multisoliton solutions in the limits (156) is similar to the single soliton solution, but f_j is replaced by $f_j^{(+)}$

$$f_j^{(+)}=\left(\frac{\tau_j}{\chi_j}\right)^{1/2}f_j, \tag{163}$$

$$\tau_j=\prod_{n=1}^{j-1}\frac{(\xi_j-\xi_n)(\xi_j+\overline{\xi_n})}{(\xi_j+\xi_n)(\xi_j-\overline{\xi_n})}, \tag{164}$$

$$\chi_j=\prod_{m=j+1}^{N}\frac{(\xi_j-\xi_m)(\xi_j+\overline{\xi_m})}{(\xi_j+\xi_m)(\xi_j-\overline{\xi_m})}. \tag{165}$$

While $\det Q\to\det Q_j^{(+)}$,

$$\det Q_j^{(+)}=-\frac{(\xi_j-\overline{\xi_j})^2}{4|\xi_j|^2|\xi_j-\overline{\xi_j}|^2}(1+|f_j^{(+)}|^{-8}) + \frac{1}{4|\xi_j|^2}[(f_j^{(+)})^{-4}+(\overline{f_j^{(+)}})^{-4}], \tag{166}$$

the asymptotic expression of $\det Q'$ should be obtained. Meanwhile, $\Phi_{1j}^{(+)}$ and $\Phi_{2j}^{(+)}$ corresponding to those in Eqs. (153)–(155) can be written as

$$\Phi_{1j}^{(+)}=\begin{cases} \dfrac{2\xi_j''(|\xi_j|^2+\rho^2)}{|\xi_j|^2}(x-V_{1j}t-x_{1j0}-\Gamma_{1j}^{(+)}) & \text{for an easy plane} \\ \dfrac{2\xi_j''(|\xi_j|^2+\rho^2)}{|\xi_j|^2}(x-V_{1j}t-x_{1j0}-\Gamma_{1j}^{(+)}) & \text{for an easy axis} \end{cases} \tag{167}$$

and

$$\Phi_{2j}^{(+)}=\begin{cases}\dfrac{2\xi_j'(|\xi_j|^2-\rho^2)}{|\xi_j|^2}(x-V_{2j}t-x_{2j0}-\Gamma_{2j}^{(+)}) & \text{for an easy plane}\\ \dfrac{2\xi_j'(|\xi_j|^2-\rho^2)}{|\xi_j|^2}(x-V_{2j}t-x_{2j0}-\Gamma_{2j}^{(+)}) & \text{for an easy axis,}\end{cases} \tag{168}$$

where

$$V_{1j}=\begin{cases}\dfrac{2\xi_j'(|\xi_j|^4+\rho^4)}{|\xi_j|^2(|\xi_j|^2+\rho^2)} & \text{for an easy plane}\\ \dfrac{4\xi_j'(|\xi_j|^2-\rho^2)}{|\xi_j|^2} & \text{for an easy axis}\end{cases} \tag{169}$$

and

$$V_{2j}=\begin{cases}\dfrac{(\xi_j'^2-\xi_j''^2)(|\xi_j|^2+\rho^2)}{\xi_j'|\xi_j|^2} & \text{for an easy plane}\\ \dfrac{2[2\rho^2|\xi_j|^4+(\xi_j'^2-\xi_j''^2)(|\xi_j|^4+\rho^4)]}{\xi_j'|\xi_j|^2(|\xi_j|^2-\rho^2)} & \text{for an easy axis,}\end{cases} \tag{170}$$

while

$$\Gamma_{1j}^{(+)}=\begin{cases}\dfrac{|\xi_j|^2}{2\xi_j''(|\xi_j|^2+\rho^2)}(\ln|\tau_j|-\ln|\chi_j|) & \text{for an easy plane}\\ \dfrac{|\xi_j|^2}{2\xi_j''(|\xi_j|^2+\rho^2)}(\ln|\tau_j|-\ln|\chi_j|) & \text{for an easy axis}\end{cases} \tag{171}$$

and

$$\Gamma_{2j}^{(+)}=\begin{cases}\arg\tau_j-\arg\chi_j & \text{for an easy plane}\\ \arg\tau_j-\arg\chi_j & \text{for an easy axis.}\end{cases} \tag{172}$$

Similarly, when $t\to-\infty$, the asymptotic behavior of multisoliton solutions in the vicinity of Θ_j can be obtained, e.g., analogously to Eqs. (171) and (172),

$$\Gamma_{1j}^{(-)}=-\Gamma_{1j}^{(+)}, \quad \Gamma_{2j}^{(-)}=-\Gamma_{2j}^{(+)}. \tag{173}$$

Therefore, the total additional displacement of Γ_{1j} and the total phase shift Γ_{2j} are

$$\Gamma_{1j}=2\Gamma_{1j}^{(+)}, \quad \Gamma_{2j}=2\Gamma_{2j}^{(+)}. \tag{174}$$

V. DISCUSSION

In the present paper, we introduce an auxiliary parameter ξ in Eqs. (25) and (26), where $\xi=\pm\rho$ corresponds to zero μ (or λ) and to $\lambda=\pm 2\rho$ (or $\mu=\pm 2\rho$). In the complex λ (or μ) plane, these two points are the edges of cuts. ξ contributes to the determination factor C_n in Eq. (48). C_n is important to ensure that the Jost solution generated satisfies the corresponding Lax equations. This indicates that in an inverse scattering transformation the edges of cuts in the complex plane must make a contribution even in the case of nonreflection. Unfortunately, Borovik and Kulinich [29,30] apparently did not consider these effects. Evidently, they did not obtain any expression of the solution.

Equations (90)–(95) show that soliton solutions in an anisotropic spin chain are dependent essentially on two velocities: V_1 in Eq. (88) and V_2 in Eq. (89). The center of an inhomogeneity moves with a constant velocity V_1, while the shape of solitary waves also changes with another velocity V_2. Therefore, the depths and widths of the surface of solitary waves are not constants but vary periodically with time, and the shape of solitary waves is not symmetrical with respect to the center. By means of these features, we find that soliton solutions in an anisotropic spin chain are not expressed in the form of a product of separated variables in moving coordinates. Only when an anisotropic parameter $\rho\to 0$ do these soliton solutions in an anisotropic spin chain reduce to those in an isotropic spin chain; for example, the single-soliton solutions (110) and (111) in the polar coordinates are equivalent to Eq. (22) obtained by means of the method of separating variables in moving coordinates in Ref. [19]. Therefore, it is impossible to investigate the exact soliton solutions in an anisotropic spin chain by means of the method of separating variables.

Using the Hirota method, Bogdan and Kovalev [31] sought the soliton solutions of the Landau-Lifshitz equation in an anisotropic spin chain in the form

$$\mathbf{S}_x+i\mathbf{S}_y=\frac{2fg}{|f|^2+|g|^2}, \quad \mathbf{S}_z=\frac{|f|^2-|g|^2}{|f|^2+|g|^2}, \tag{175}$$

where

$$f=\sum_{n=0}^{[N/2]}\sum_{C_2,n}a(i_1,\ldots,i_{2n})\exp(\rho_{i_1}+\cdots+\rho_{i_{2n}}) \tag{176}$$

and

$$g^*=\sum_{m=0}^{[(N-1)/2]}\sum_{C_{2m+1}}a(j_1,\ldots,j_{2m+1})\times\exp(\rho_{j_1}+\cdots+\rho_{j_{2m+1}}), \tag{177}$$

where $[N/2]$ is the maximum integer in addition to $N/2$, C_n represents the summation over all combinations of N elements in n, and $\rho_i=(k_i+\omega_i t+\rho_i^0)$, while

$$a(i_1,\ldots,i_n)=\begin{cases}\sum_{k<l}^{(n)}a(i_k,i_l) & \text{for } n\geq 2\\ 1 & \text{for } n=0,1.\end{cases} \tag{178}$$

According to the expression of the single-soliton solutions (90)–(95) in this paper, we find that they are difficult to express in the form of Hirota factorization. Obviously, Bogdan and Kovalev [31] did not obtain the desired results.

Reducing the equations of motion to an appropriate form, Kosevich *et al.* [25] found a solution in the classical Heisenberg spin chain with an easy plane, while in terms of Eq. (96) in the polar coordinates in the present paper, there exists

$$\tan^2\left(\frac{\theta}{2}\right)=\frac{\xi_1''^2\left\{(|\xi_1|^2-\rho^2)^2+4\rho^2|\xi_1|^2\sin^2\left[\frac{2\xi_1'(|\xi_1|^2-\rho^2)}{|\xi_1|^2}(x-V_2t-x_{20})\right]\right\}}{(|\xi_1|^2-\rho^2)^2\left\{|\xi_1|^2\cosh^2\left[\frac{2\xi_1''(|\xi_1|^2+\rho^2)}{|\xi_1|^2}(x-V_1t-x_{10})\right]-\xi_1''^2(|\xi_1|^2-\rho^2)^2\right\}}. \tag{179}$$

If we compare Eq. (179) with an approximate solution given by Ref. [25], we find that previous properties of the soliton solutions remain even in the approximation of order of ρ^2. The solutions of Ref. [25] do not satisfy the Landau-Lifshitz equation for the classical Heisenberg spin chain with an easy plane even in the first order of anisotropy, and there is no reason to consider it as an approximate solution; all attempts in this approximation were not successful.

In the previous discussion using the suitable rescaling and an appropriate spin Poisson bracket, the equations of motion are obtained for an anisotropic Heisenberg spin chain with Gilbert damping in an external magnetic field. Then, introducing an auxiliary parameter, the Lax equations for Darboux matrices are generated recursively. If constants are suitably chosen, the Jost solutions satisfy the corresponding Lax equations. The exact soliton solutions are obtained; then the asymptotic behavior of the multisoliton solutions is investigated and the z components of the total magnetic momentum, and the total magnetic momentum, are given. These results have not previously been found, to our knowledge, by any means tried. They may be useful for further theoretical research and practical applications.

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